Study of the effective stability in the restricted three body problem

Ch. Skokos[1], *G. Contopoulos*[1] *and A. Giorgilli*[2]

University of Athens, Department of Physics,
Section of Astrophysics, Astronomy and Mechanics,
Panepistimiopolis, GR-157 83 Zografos, Athens, Greece

[2] Dipartimento di Fisica dell' Universita di Milano,
Via Celoria 16, Milano, Italy

Abstract

We consider a new coordinate system in the Sun–Jupiter–asteroid problem, that allows for an improved estimate of the confinement regions around L_4 and L_5. We prove that stability over the age of the universe is guaranteed in a region larger than any previous estimate.

1 Introduction

We try to find a region of *effective stability* around the Lagrangian points L_4 and L_5 in the *planar restricted problem of three bodies*. That means that orbits with initial conditions in a suitable neighbourhood of the equilibrium, are confined to a slightly larger neighbourhood for a very long time interval. For example in the case of Sun, Jupiter and asteroid, this time can be as large as the age of the universe. We investigate the problem combining analytical methods and numerical approximations, trying to produce a realistic estimate of the stability region.

2 The Hamiltonian

The planar restricted three body problem is well known [1]. As usual we consider a body A of infinitesimal mass (asteroid), orbiting in the gravitational field of the two primaries S, J with masses respectively equal to $1-\mu$ (Sun) and μ (Jupiter), which are assumed to revolve in circular orbits around their common centre of mass. We introduce a uniformly rotating frame (O, q_1, q_2) as follows : the origin is located at the center of mass, the Sun is always at the point $(\mu, 0)$ and Jupiter at the point $(\mu-1, 0)$, assuming that the gravitational constant is one and taking as unit of length the distance between the Sun and Jupiter. Moreover $\mu = 9.5387536\,10^{-4}$. Then the Hamiltonian is of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 - \frac{1-\mu}{\sqrt{(q_1-\mu)^2 + q_2^2}} - \frac{\mu}{\sqrt{(q_1+1-\mu)^2 + q_2^2}} \tag{1}$$

We now perform some canonical transformations on the Hamiltonian. Firstly we introduce a rotating frame (O, Q_1, Q_2), with its origin on the Sun, thus changing the Hamiltonian to

$$H = \frac{1}{2}(P_1^2 + P_2^2) + Q_2P_1 - Q_1P_2 - \mu Q_1 - \frac{1-\mu}{\sqrt{Q_1^2+Q_2^2}} - \frac{\mu}{\sqrt{(Q_1+1)^2+Q_2^2}} - \frac{\mu^2}{2} \quad (2)$$

Secondly, we introduce polar variables Q_1', Q_2' via $Q_1 = Q_1'\cos(Q_2')$, $Q_2 = Q_2'\cos(Q_2')$, and change the Hamiltonian as

$$H = \frac{1}{2}(P_1'^2 + \frac{P_2'^2}{Q_1'^2}) - P_2' - \mu Q_1'\cos(Q_2') - \frac{1-\mu}{Q_1'} - \frac{\mu}{\sqrt{Q_1'^2 + 1 + 2Q_1'\cos(Q_2')}} \quad (3)$$

The coordinates of L_4 are $Q_1' = 1, Q_2' = \frac{2\pi}{3}, P_1' = 0, P_2' = 1$. Thirdly, we perform the transformation $x = Q_1' - 1, y = Q_2' - \frac{2\pi}{3}, p_x = P_1', p_y = P_2' - 1$, which corresponds to moving the origin to the point L_4, and transform the Hamiltonian to

$$\left.\begin{array}{l} H = \frac{1}{2}[p_x^2 + \frac{(p_y+1)^2}{(x+1)^2}] - p_y - \mu(x+1)\cos(y+\frac{2\pi}{3}) - \frac{1-\mu}{x+1} - \\ \quad - \frac{\mu}{\sqrt{(x+1)^2+1+2(x+1)\cos(y+\frac{2\pi}{3})}} \end{array}\right\} \quad (4)$$

Following the methods used for studying the dynamics in the neighbourhood of an equilibrium, we expand H in power series as : $H = \sum_{j=0}^{\infty} H_j$ where $H_0 = \frac{\mu-1}{2}$, $H_1 = 0$ and

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) - 2xp_y + (\frac{1}{2} + \frac{9\mu}{8})x^2 - \frac{9\mu}{8}y^2 + \frac{3\sqrt{3}\,\mu}{4}xy \quad (5)$$

The next step is a change of variables from (x, y, p_x, p_y) to (x_1, x_2, y_1, y_2), such that H_2 takes the diagonal form $\frac{1}{2}\sum_{j=1}^{2}\omega_j(x_j^2 + y_j^2)$. Following previous papers [2],[3] we introduce the parameter $\alpha = -\frac{(1-2\mu)3\sqrt{3}}{4}$. By denoting the eigenvalues of the matrix $\mathbf{A} = \mathbf{J}\,\mathbf{S}$ as $\lambda = i\omega$,where $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is the usual symplectic matrix and $\mathbf{S}$ is the Hessian of H_2, we find the characteristic equation

$$\omega^4 - \omega^2 + \frac{27}{16} - \alpha^2 = 0 \quad (6)$$

Since $27\mu(1-\mu) < 1$ we get two positive solutions:

$$\omega_1^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2}\,,\ \omega_2^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2} \quad (7)$$

where $\omega_1^2 > \frac{1}{2}, \omega_2^2 < \frac{1}{2}$.

For every eigenvalue $i\omega_j, j = 1, 2$ the corresponding eigenvector is :

$$b_j = \begin{pmatrix} \frac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8} \\ \frac{16i\omega_j + 4\alpha + 3\sqrt{3}}{8} \\ i\omega_j(\frac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8}) \\ i\omega_j\frac{4\alpha + 3\sqrt{3}}{8} + \frac{4\sqrt{3}\alpha + 9}{4} \end{pmatrix} \quad (8)$$

Writing $b_j = e_j + i f_j$ with e_j and f_j real vectors, by a simple computation we get $e_j^T \mathbf{J} f_j = m_j = \omega_j D_j$ with $D_j = (\frac{8\omega_j^2+4\sqrt{3}\alpha+9}{8})^2 - 2(\sqrt{3}\alpha + \frac{9}{4}) + (\frac{4\alpha+3\sqrt{3}}{8})^2$ which is positive for $j = 1$ and negative for $j = 2$.

The matrix $C = (e_1 m_1^{-\frac{1}{2}}, e_2 m_2^{-\frac{1}{2}}, f_1 m_1^{-\frac{1}{2}}, f_2 m_2^{-\frac{1}{2}})$ is symplectic and one has $\mathbf{C}^T \mathbf{S}\, \mathbf{C} = \mathrm{diag}\,(\omega_1, \omega_2, \omega_1, \omega_2)$. ¿From the reality of $\mathbf{C}$ we get $m_j > 0$ $j = 1,2$ and therefore $\omega_1 > 0$, $\omega_2 < 0$. $\mathbf{C}$ is the transformation matrix so that $(x, y, p_x, p_y)^T = \mathbf{C}\,(x_1, x_2, y_1, y_2)^T$ and the Hamiltonian in the new variables has the form $H = \sum_{j=2}^{\infty} H_j$ with

$$H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) \tag{9}$$

3 The normal form

The procedure for finding the normal form is explained in detail in Giorgilli et al. [2]. Here we will mention the basic ideas. Using the canonical transformation $q_j = \frac{1}{\sqrt{2}}(x_j - i\, y_j)$, $p_j = -\frac{i}{\sqrt{2}}(x_j + i\, y_j)$, $j = 1,2$ we write the Hamiltonian in complex variables (q_1, q_2, p_1, p_2). The second order of the Hamiltonian takes the form $H_2 = i\,\omega_1 p_1 q_1 + i\,\omega_2 p_2 q_2$. Then we transform the Hamiltonian to a Birkhoff normal form. To this end, considering the power series expansion of the Hamiltonian up to order r in complex variables, namely $H = H_2 + H_3 + \ldots + H_r$, we look for a sequence $X_1 + X_2 + \ldots + X_r$ of generating functions by solving the equation

$$H \;=\; T_X\, Z \tag{10}$$

with the condition that the transformed Hamiltonian $Z = Z_2 + Z_3 + \ldots + Z_r$ is in Birkhoff normal form.

In order to define how the operator T_X acts let E be the space of the formal series in complex variables $(q_1, q_2, p_1, p_2) \in C^4$ and E_k the subspace of the homogeneous polynomials of degree k, so $f_k \in E_k$ if $f_k = \sum_{|l+m|=k} f_{l,m}\, q^l p^m$ where $q^l p^m = q_1^{l_1} q_2^{l_2} p_1^{m_1} p_2^{m_2}$ with $l_1, l_2, m_1, m_1 \in N$ and $f_{l,m} \in C$. If $g \in E_l, f \in E_k$ we define an operator $L_g : E_k \to E_{k+l-2}$ so that $L_{gf} = \{g, f\}$ is the Poisson bracket. So if $f = \sum_{k\geq 1} f_k$, we have

$$T_X f \;=\; \sum_{k\geq 1} F_k \tag{11}$$

where

$$F_k \;=\; \sum_{l=1}^{k} f_{l,k-l} \tag{12}$$

for

$$f_{l,0} \;=\; f_l \quad , \quad f_{l,k} = -\sum_{m=1}^{k} \frac{m}{k}\, L_{X_{2+m}}\, f_{l,k-m} \tag{13}$$

So we understand the meaning of (10) by replacing the function f of the relations (11) , (12) and (13) by Z. The operator T_X turns out to be invertible. An explicit algorithm for the computation of the inverse is given, e.g., in [4].

4 The region of effective stability

As we already explained having the expansion of H up to order r we find the normal form Z up to this order, thus

$$T_X^{-1} H = \underbrace{Z_2 + Z_3 + \cdots + Z_r}_{normal\ form} + \underbrace{Y^{r+1} + Y^{r+2} + \cdots}_{remainder} \tag{14}$$

The normal form has two exact integrals:

$$I_l = \frac{1}{2}(x_l'^2 + y_l'^2)\ l = 1,2 \quad , \quad \{I_l, Z_2 + Z_3 + \ldots + Z_r\} = 0 \tag{15}$$

Of course I_1, I_2 transformed to x_1, x_2, y_1, y_2 variables are not exact integrals of the Hamiltonian because of the remainder in (14). Assuming that the first order of the remainder gives the most important contribution we get

$$\dot{I}_l \approx \{I_l, Y^{r+1}\} \tag{16}$$

Following Celletti and Giorgilli [3], we introduce a norm as follows. For a homogeneous polynomial of order s $f(x', y') = \sum_{j,k} f_{jk}\, x'^j y'^k$ where $x'^j y'^k = x_1'^{j_1} x_2'^{j_2} y_1'^{k_1} y_2'^{k_2}$ with complex coefficients f_{jk} in general, we define

$$\|f\|_R = \sum_{j,k} |f_{jk}| R^{j+k} \qquad (R^{j+k} = R_1^{j_1+k_1}\, R_2^{j_2+k_2}) \tag{17}$$

where R_1, $R_2 \in R_+^*$. Now we consider the domain

$$\Delta_{\rho R} = \{(x', y') \in R^4 : x_l'^2 + y_l'^2 \le \rho^2 R_l^2 \quad (l = 1,2)\} \tag{18}$$

So for any homogeneous polynomial of order s we get

$$|f(x', y')| \le \rho^s \|f\|_R \tag{19}$$

Also we have

$$(x', y') \in \Delta_{\rho R} \Rightarrow I_l \le \frac{1}{2} R_l^2 \rho^2 \quad (l = 1,2) \tag{20}$$

Using (16) and (19) we get

$$|\dot{I}_l| \le \|\{I_l, Y^{r+1}\}\|_R\, \rho^{r+1} = C_{l,r}\rho^{r+1} \tag{21}$$

Assume that the initial point of an orbit lies in the domain $\Delta_{\rho_0 R}$ and that we ask the orbit to be confined inside a domain $\Delta_{\rho R}$ with $\rho > \rho_0$, for a finite time interval. We shall refer to this time interval as the escape time τ. By the definition of the domains, it is enough to ask $|I_l(t) - I_l(0)| \approx |\dot{I}_l|\, \tau \le \frac{1}{2} R_l^2(\rho^2 - \rho_0^2)$. Thus using (21) we find the escape time $\tau_{l,r}(\rho_0, \rho)$, as a function of l, r, ρ and ρ_0.

$$\tau_{l,r}(\rho_0, \rho) = \frac{R_l^2(\rho^2 - \rho_0^2)}{2C_{l,r}\rho^{r+1}} \tag{22}$$

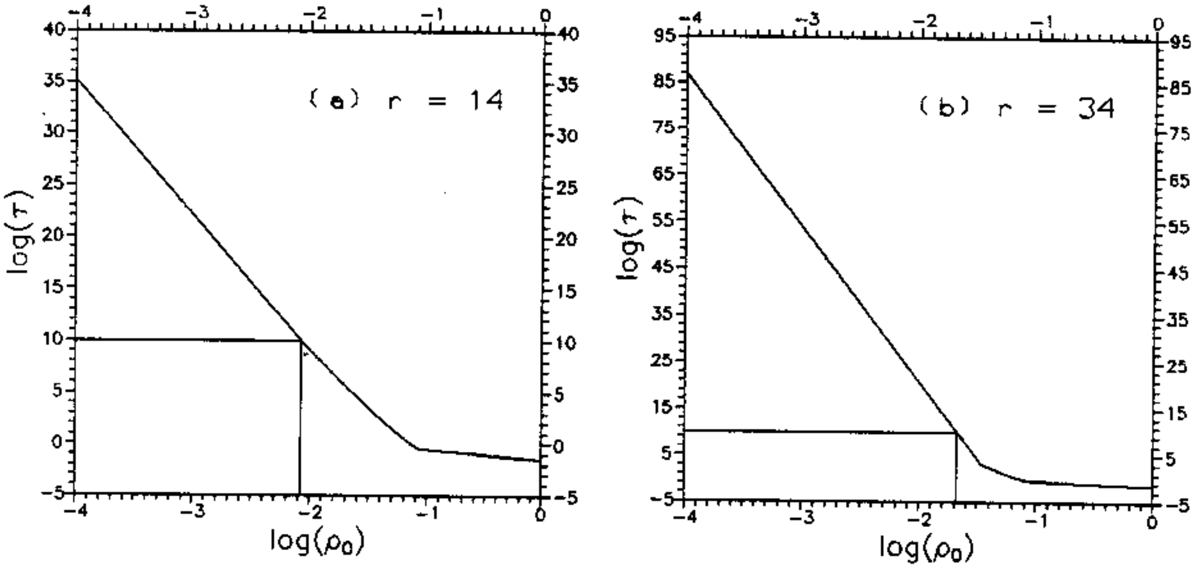

Figure 1: The maximum escape time $\tau(\rho_0)$ as a function of the radius ρ_0 of the initial domain $\Delta_{\rho_0 R}$, for the cases of finding the normal form up to order 14 (a) and 34 (b). We mark the value of ρ_0 corresponding to time equal to the age of the universe $T_{un} = 10^{10}$.

We can optimize the escape time against l, r and ρ and get it as a function of the initial domain $\Delta_{\rho_0 R}$. The escape time becomes maximum with respect to ρ when $\frac{\rho^2-\rho_0^2}{\rho^{r+1}}$ becomes maximum. That happens for $\rho = \rho_* = \rho_0\sqrt{\frac{r+1}{r-1}}$. That means that we overestimate the value of $|\dot{I}_l|$ by considering it equal to its value at the limit of the domain $\Delta_{\rho R}$. For every r we compute the quantity $A_{l,r} = \frac{R_l^2}{2C_{l,r}}$ for $l = 1, 2$ and we keep the smallest one, because if the orbit is outside the projections of the domain on the plane (x_1', y_1') or (x_2', y_2') it is also outside $\Delta_{\rho R}$. Thus we get $A_r = \min_l A_{l,r}$. Then by putting ρ_* and A_r in (22) we compute the escape time for all the values of r. We define the optimal order of the expansion r_{opt} as the one that gives the maximum escape time. Thus we get the escape time τ as a function of ρ_0, $\tau(\rho_0)$ because we put $R_1 = R_2 = 1$. In the units used here the age of the universe is about $T_{un} = 10^{10}$. So we can define the value of ρ_0 for which the time needed for an orbit to escape is 10^{10}. By doing the expansion of the Hamiltonian up to some order r, and finding the normal form up to $r-1$, we have the remainder starting from order r, and then we estimate the region of effective stability $\Delta_{\rho_0 R}$. In FIG.1 we plot the escape time $\tau(\rho_0)$ with respect to ρ_0 when the normal form is up to order $r = 14$ and $r = 34$. Thus we find

$$\begin{array}{lllll} r = 14 & \log(\rho_0) \approx -2.08 & \Rightarrow & \rho_0 \approx 8.31\,10^{-3} \\ r = 34 & \log(\rho_0) \approx -1.67 & \Rightarrow & \rho_0 \approx 2.14\,10^{-2} \end{array} \tag{23}$$

Celletti and Giorgilli [3] in a similar analysis have found $\rho_0 \approx 2.58\,10^{-4}$. The results can be compared because we use the same units. We see that we improved the previous result by a factor 80.

5 Conclusions

We found the region of effective stability in the planar restricted three body problem, around the point L_4, by writing the Hamiltonian of the system in appropriate variables and using both analytical methods and numerical approximations. Thus we improved older estimates by a factor 80. Some efforts for improving further this result will be made in the near future.

References

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